DIFFERENTIAL EQUATIONS OF HEAT CONDUCTION IN A DISPERSE SYSTEM

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A system of differential equations is proposed for the description of unsteady heat conduction in disperse systems. A solution is derived for boundary conditions of the 1st, 2nd, and 3rd kind.

It is well known that in the theory of heat conduction and heat transfer in disperse systems, in addition to the different model representations that take account of the dispersion of the medium [1, 2], there is a tendency to use the ideas and the mathematical apparatus of the theory of single-phase continua [3–6].

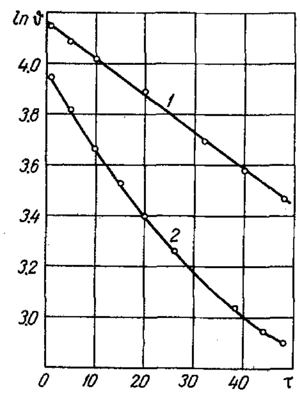

Fig. 1. Temperature of plate surface: 1) when cooled in air; 2) when cooled in a bed (τ in sec).

The first route has led to a series of simplified relations which permit a qualitative estimate of heat transfer, while the second is also used to obtain design relations.

Harakas and Beatty [7] have shown that in the region from steady heat transfer up to a certain value Fo_{min} (which has been determined only in isolated cases), the use of the heat conduction differential equation for a continuum gives results in agreement with experimental data.

For values of $Fo < Fo_{min}$, the deviation between experimental data and theory becomes larger as Fo diminishes. To obtain agreement between experimental and theoretical data when $\tau \to 0$, use was made of the idea proposed earlier in [1] that there exists a gas sublayer between the heat transfer surface and the first row of particles, the thermal resistance of the sublayer being assumed to be approximately independent of time. Then the heat flux at any time is determined by the sum of the thermal resistances:

$$R_\kappa = R_{(\lambda)} + R_{(\tau)}.$$

In the differential equation of heat conduction for a continuum the value of thermal conductivity used is that measured under steady conditions (λ_{eff}). When $\tau \to 0$ $R_{(\tau)}$ tends to zero for the continuum for this choice of the value of $R_{(\lambda)}$ we may secure agreement between the experimental and theoretical heat flux values by choice of the thickness δ of the gas sublayer. By the use of this kind of artificial means we may achieve some measure of agreement between experiment and theory, but physically this method is poorly based. It is difficult to answer a number of questions, namely: why is it only the first row of particles that is allotted the thermal resistance $R_{(\lambda)}$, why is it considered that the thermal resistance of the sublayer is steady, and whether there is not a considerable discrepancy, nevertheless, between theory and experiment when we attempt, by this arbitrary means, to secure agreement at a certain τ between experimental data and theory by choice of some δ, and so on.

There has been an attempt to allot all the particles in the layer to the thermal resistance [4], but then the approximate heat transfer model is illustrated by an approximate solution of the system of equations, as a result of which the reliability of the relations obtained is insufficient for a complete qualitative and quantitative analysis.

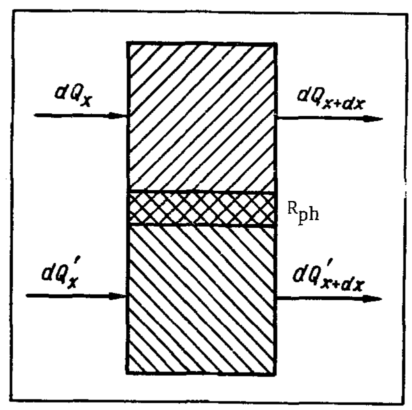

Fig. 2. Model of the volume element for a disperse system.

Figure 1 shows results of measurement of cooling rate of a copper plate measuring 100 × 45 × 1 mm in a bed of 3-mm particles of lead glass and in air; the results show that, in contrast with the case for a continuum, the heat transfer coefficient depends on time, and, consequently, Newton's law of cooling cannot be applied to the disperse medium as a whole. The divergence of the test data from theory at small Fo [7] also does not permit the Fourier law to be applied to a disperse medium. However, since both the Fourier law and Newton's law undoubtedly are valid for each of the "continua" individually, we shall use them for derivation of the differential equations of heat conduction of the system.

We shall examine two volume elements (Fig. 2), separated by an unknown thermal resistance. For each volume element, in the condition $R_{ph} \to \infty$, for a one-dimensional problem, the relations

$$\frac{\partial t_1}{\partial \tau} = a_1 \frac{\partial^2 t_1}{\partial x^2}, \quad (1)$$

$$\frac{\partial t_2}{\partial \tau} = a_2 \frac{\partial^2 t_2}{\partial x^2} \quad (2)$$

are valid.

In the case when a_1 and a_2 are equal, $t_1 = t_2$, and for any R_{ph} the system may be regarded as a homogeneous continuum. In all other cases it should be supposed that the component volume element has two temperatures (t_1 and t_2). In this case we may write

$$\frac{\partial t_1}{\partial \tau} = a_1 \frac{\partial^2 t_1}{\partial x^2} + \beta_1 (t_2 - t_1), \quad (3)$$

$$\frac{\partial t_2}{\partial \tau} = a_2 \frac{\partial^2 t_2}{\partial x^2} - \beta_2 (t_2 - t_1). \quad (4)$$

It is then assumed that heat transfer between phases follows Newton's law ($\beta_n = \alpha S/(c\gamma)_n$), an assumption that places a restriction on Eqs. (3) and (4), but this restriction will evidently not be appreciable when we are considering finely dispersed systems. Thus, for heat conduction in a bed, the mutual influence of the phases manifests itself in the fact that the medium with the smaller thermal diffusivity slows down the development of the temperature field of the medium with the larger diffusivity, the degree of interaction of the media being determined by the difference in the diffusivities of the media and by the intensity of heat transfer between the phases. Thus, for example, if the diameter of the particles of the disperse phase of the material is very small, the link between the temperature fields is large, so that in the limit when $d \to 0$, $\alpha S \to \infty$ and $\Delta t = (t_2 - t_1) \to 0$ the disperse material may be considered continuous. With increase of particle diameter and increase in the difference of the diffusivity of the phases, on the other hand, Δt increases, which allows the use of Eqs. (3) and (4) to describe the heat conduction in a disperse system, with good justification.

A solution is given below of the system of differential Eqs. (3) and (4) with the following initial and boundary conditions:

$$t_1|_{\tau=0} = t_2|_{\tau=0} = t_0, \quad (5)$$

$$\left(A_1 \frac{\partial t_1}{\partial x} + B_1 t_1 + C_1 \right)\Big|_{x=0} = 0, \quad (6)$$

$$\left(A_2 \frac{\partial t_2}{\partial x} + B_2 t_2 + C_2 \right)\Big|_{x=0} = 0, \quad (7)$$

$$t_1|_{x=R} = t_2|_{x=R} = t_0. \quad (8)$$

We shall seek a solution in the form of the sum of the steady and unsteady solutions, i.e.,

$$t_1(x, \tau) = \bar{t}_1(x) + v_1(x, \tau), \quad (9)$$

$$t_2(x, \tau) = \bar{t}_2(x) + v_2(x, \tau). \quad (10)$$

Assuming in Eqs. (6) and (7) that $A_1 = A_2 = 0$, $B_1 = B_2 = 1$, $C_1 = C_2 = -t_w$, and applying a finite Fourier sine transformation with respect to x and a Laplace transformation with respect to τ to Eqs. (3) and (4) in succession, we obtain the following expressions for the temperature field in the gas and solid phases:

$$t_1(x, \tau) = \frac{t_0 x + t_w (R - x)}{R} + \frac{2(t_0 - t_w)}{R} \sum_{n=1}^{\infty} \frac{1}{\mu_n (P_2 - P_1)} \times \left[(D_n + P_2) \exp(P_2 \tau) - (D_n + P_1) \exp(P_1 \tau) \right] \sin(\mu_n x), \quad (11)$$

$$t_2(x, \tau) = \frac{t_0 x + t_w (R - x)}{R} + \frac{2(t_0 - t_w)}{R} \sum_{n=1}^{\infty} \frac{\beta_2}{\mu_n (P_2 - P_1)} \times \left[\frac{D_n + P_2}{A_{n(2)} + P_2} \exp(P_2 \tau) - \frac{D_n + P_1}{A_{n(2)} + P_1} \exp(P_1 \tau) \right] \sin(\mu_n x), \quad (12)$$

where μ_n are the roots of the characteristic equation

$$\sin(\mu_n R) = 0 \quad (13)$$

and

$$P_1 = \frac{-B_n + \sqrt{B_n^2 - 4C_n}}{2}, \quad P_2 = \frac{-B_n - \sqrt{B_n^2 - 4C_n}}{2},$$

$$B_n = A_{n(1)} + A_{n(2)}, \quad A_{n(1)} = a_1 \mu_n^2 + \beta_1, \quad A_{n(2)} = a_2 \mu_n^2 + \beta_2,$$

$$C_n = A_{n(1)} \cdot A_{n(2)} - \beta_1 \cdot \beta_2, \quad D_n = A_{n(2)} + \beta_1.$$

For the case $a_1 = 0$ and $t_0 = 0$ the problem is simplified considerably. A detailed solution for this case has been given in [8].

In a similar way we shall obtain a solution of the system (3) and (4) for boundary conditions of the second kind, by putting $A_1 = \lambda_1$, $A_2 = \lambda_2$, $B_1 = B_2 = 0$, $C_1 = q_1$, $C_2 = q_2$ and applying successively a Fourier cosine transformation with respect to x and a Laplace transformation with respect to τ, i.e.,

$$t_1(x, \tau) = t_0 + \frac{q_1}{\lambda_1}(R - x) - \frac{2q_1}{R\lambda_1} \sum_{n=1}^{\infty} \frac{1}{\mu_n^2 (P_2 - P_1)} \times [(P_2 + C_n) \exp(P_2 \tau) - (P_1 + C_n) \exp(P_1 \tau)] \cos(\mu_n x), \quad (14)$$

$$t_2(x, \tau) = t_0 + \frac{q_2}{\lambda_2}(R - x) - \frac{2q_2}{R\lambda_2} \sum_{n=1}^{\infty} \frac{\beta_2}{\mu_n^2 (P_2 - P_1)} \times \left[\frac{P_2 + C_n}{P_2 + D_n} \exp(P_2 \tau) - \frac{P_1 + C_n}{P_1 + D_n} \exp(P_1 \tau) \right] \cos(\mu_n x), \quad (15)$$

where μ_n are the roots of the characteristic equation

$$\cos(\mu_n R) = 0 \quad (16)$$

and

$$P_1 = \frac{-A_n + \sqrt{A_n^2 - 4B_n}}{2},$$

$$P_2 = \frac{-A_n - \sqrt{A_n^2 - 4B_n}}{2},$$

$$A_n = (a_1 \mu_n^2 + \beta_1) + (a_2 \mu_n^2 + \beta_2),$$

$$B_n = (a_1 \mu_n^2 + \beta_1)(a_2 \mu_n^2 + \beta_2) - \beta_1 \beta_2,$$

$$C_n = a_2 \mu_n^2 + \beta_1 + \beta_2, \quad D_n = a_2 \mu_n^2 + \beta_2.$$

To solve the problem with boundary conditions of the 3rd kind, we must put $A_1 = B_1 = 0$; $A_2 = 1$;

$$B_2 = -\frac{\alpha}{\lambda_2}; \quad C_2 = \frac{\alpha}{\lambda_2} t_m$$

in Eqs. (6) and (7).

Carrying out the appropriate integral transformations, as above, we obtain the solution in the form

$$t_1(x, \tau) = t_m + \frac{t_0 - t_m}{\lambda_2 + \alpha R}(\lambda_2 + \alpha x) + 2\sum_{n=1}^{\infty} \frac{\alpha (t_0 - t_m)}{\lambda_2 \mu_n} \times$$

$$\times \frac{[\mu_n \cos \mu_n x + (\alpha/\lambda_2) \sin \mu_n x]}{[\alpha/\lambda_2 + (\mu_n^2 + \alpha^2/\lambda_2^2) R]} \frac{\beta_1}{P_2 - P_1} \times$$

$$\times \left[\frac{P_2 + \beta_1 + \beta_2}{P_2 + \beta_1} \exp(P_2 \tau) - \right.$$

$$\left. - \frac{P_1 + \beta_1 + \beta_2}{P_1 + \beta_1} \exp(P_1 \tau) \right], \tag{17}$$

$$t_2(x, \tau) = t_m + \frac{t_0 - t_m}{\lambda_2 + \alpha R}(\lambda_2 + \alpha x) + 2\sum_{n=1}^{\infty} \frac{\alpha (t_0 - t_m)}{\lambda_2 \mu_n} \times$$

$$\times \frac{[\mu_n \cos \mu_n x + (\alpha/\lambda_2) \sin \mu_n x]}{[\alpha/\lambda_2 + (\mu_n^2 + \alpha^2/\lambda_2^2) R]} \left[\frac{P_2 + \beta_1 + \beta_2}{P_2 - P_1} \exp(P_2 \tau) - \right.$$

$$\left. - \frac{P_1 + \beta_1 + \beta_2}{P_2 - P_1} \exp(P_1 \tau) \right], \tag{18}$$

where μ_n are the roots of the characteristic equation

$$\mu_n \cos(\mu_n R) + \frac{\alpha}{\lambda_2} \sin(\mu_n R) = 0, \tag{19}$$

and

$$P_1 = \frac{-A_n + \sqrt{A_n^2 - 4B_n}}{2}, \quad P_2 = \frac{-A_n - \sqrt{A_n^2 - 4B_n}}{2},$$

$$A_n = a_2 \mu_n^2 + \beta_1 + \beta_2, \quad B_n = a_2 \mu_n^2 \beta_1.$$

NOTATION

R_K is the total thermal resistance of bed; $R_{(\lambda)}$ is the thermal resistance of gas sublayer between first row of particles and heat transfer surface, this resistance being independent of time; $R_{(\tau)}$ is the thermal resistance of bed; R_{ph} is the thermal resistance to heat transfer between phases; λ_{eff} is the effective thermal conductivity of bed; δ is the thickness of gaseous sublayer; S is the surface area of particles in unit volume; α is the coefficient of heat transfer between particles and gas; α_1 is the coefficient of heat transfer between bed and medium; d is the diameter of particles; R is the interval length; t_1 is the temperature of solid; t_2 is the temperature of gas; t_0 is the initial temperature of bed; t_w is the wall temperature (at the boundary x = 0); t_m is the temperature of medium; ϑ is the excess temperature of plate surface; λ_1 is the thermal conductivity of bed; λ_2 is the thermal conductivity of medium; τ is the time.

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